Homework 2

PSTAT 5A: Spring 2023, with Ethan P. Marzban

Instructions

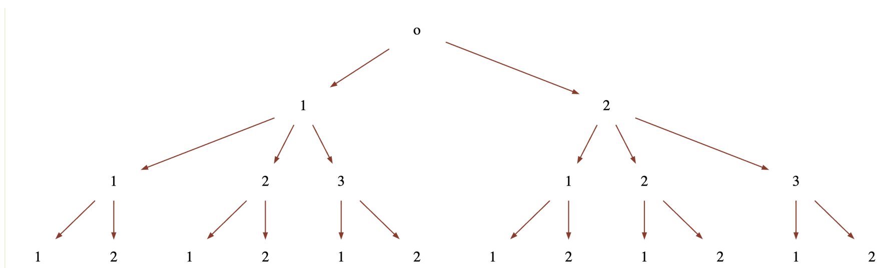
- Please submit your work to Gradescope by no later than **11:59pm on Tuesday, April 18**. As a reminder, late homework will not be accepted.
- Recall that you will be asked to upload a **single** PDF containing your work for *both* the programming and non-programming questions to Gradescope.
  - You can merge PDF files using either Adobe Acrobat, or using adobe's online PDF merger at this link.

Problem 1: Picking Numbers

A random number generator picks a number from the set $\{1, 2\}$ at random, then picks another number from the set $\{1, 2, 3\}$ at random, and finally picks a third number from the set $\{1, 2\}$ at random. The number selected at each stage is recorded.

a) Use a tree diagram to specify the outcome space of this experiment.

Solution:



b) Are we justified in using the classical approach to probability? Why or why not?

Solution: Yes, becuase the three numbers are selected *at random*.

c) Use the classical approach to probability to compute the probabilities of the following events (being sure to use proper notation!):

i. E = "the first number selected is the number 1"

Solution: The outcome space contains 12 elements. Of these 12 outcomes, there are only 6 in which the first number selected is 1:

$$E = \{(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,3,1),\ (1,3,2)\}$$

Hence, by the classical appraoch to probability,

$$\mathbb{P}(E) = \frac{6}{12} = \boxed{\frac{1}{2} = 50\%}$$

ii. F = "the second number selected is the number 2"

Solution: Of the 12 outcomes in Ω, there are only 4 in which the second number selected is 1:

$$E = \{(1,2,1),\ (1,2,2),\ (2,2,1),\ (2,2,2)\}$$

Hence, by the classical appraoch to probability,

$$\mathbb{P}(F) = \frac{4}{12} = \boxed{\frac{1}{3} = 33.\overline{3}\%}$$

iii. G = "either the first number selected is the number 1 or the second number selected is the number 2 (or both)"

Solution: Note that the event G is equivalent to $E \cup F$, where E and F are defined as in parts (i) and (ii) above. As such, we can apply the **addition rule** to compute

$$\mathbb{P}(G) = \mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$$

We computed $\mathbb{P}(E)$ and $\mathbb{P}(F)$ in the previous parts; as such, we only need to compute $\mathbb{P}(E \cap F)$. Note that the event $E \cap F$ is equivalent to "the first number selected was ‘1‘ and the second number selected was ‘2‘"; there are only 2 outcomes in Ω that satisfy this:

$$E \cap F = \{(1,2,1),\ (1,2,2)\}$$

meaning, by the classical approach to probability,

$$\mathbb{P}(E \cap F) = \frac{2}{12} = \frac{1}{6}$$

and so

$$\begin{aligned} \mathbb{P}(G) = \mathbb{P}(E \cup F) &= \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \boxed{\frac{2}{3} = 66.\overline{6}\%} \end{aligned}$$

d) Compute the probability that the sum of the last two numbers selected is strictly greater than the first number. **Hint:** Remember the complement rule!

Solution: Admittedly, this part requires a bit of thought. Following the hint, we let A denote the event ”the sum of the last two numbers is strictly greater than the first number” and examine

$$A^{\complement} = \text{“the sum of the last two numbers is less than or equal to the first number”}$$

The smallest sum we can obtain from the last two numbers is 2 (i.e. when the last two numbers selected were both 1), which reveals to us that there is only one outcome in $A^{\complement}$:

$$A^{\complement} = \{(2, 1, 1)\}$$

Hence, by the classical approach to probability,

$$\mathbb{P}(A^{\complement}) = \frac{1}{12}$$

and so by the **complement rule** we have

$$\mathbb{P}(A) = 1 - \mathbb{P}(A^{\complement}) = 1 - \frac{1}{12} = \boxed{\frac{11}{12}}$$

Problem 2: At the Movies

A recent survey at local cinemas revealed that, of the moviegoers surveyed, 75% purchase popcorn, 60% purchase a drink.

a. Is it possible for the events P = “a randomly selected moviegoer purchases popcorn” and D = “a randomly selected moviegoer purchases a drink” to be disjoint? Why or why not?

Solution: No , it is not possible. The explanation is as follows: if P and D were disjoint, then, by definition, $P \cap D = \varnothing$ and so $\mathbb{P}(P \cap D) = 0$. However, by the addition rule, this would mean $\mathbb{P}(P \cup D) = \mathbb{P}(P) + \mathbb{P}(D) = 0.75 + 0.6 = 1.35$, which is impossible since probabilities need to be less than 1! As such, our original assumption that P and D were disjoint must be false.

Now, suppose that it is also known that 40% of moviegoers purchase both popcorn and a drink.

b. What is the probability that a randomly selected moviegoer purchases *neither* popcorn *nor* a drink? **Hint:** Think about complements.

Solution: We seek the quantity $\mathbb{P}(P^{\complement} \cap D^{\complement})$. Following the hint, we recall one of **DeMorgan's Laws** which states

$$(P^{\complement} \cap D^{\complement}) = (P \cup D)^{\complement}$$

As such, we first compute $\mathbb{P}(P \cup D)$ using the **addition rule:**

$$\mathbb{P}(P \cup D) = \mathbb{P}(P) + \mathbb{P}(D) - \mathbb{P}(P \cap D) = 0.75 + 0.6 - 0.4 = 0.95$$

Hence, by the **complement rule**,

$$\mathbb{P}(P^{\complement} \cap D^{\complement}) = \mathbb{P}[(P \cup D)^{\complement}] = 1 - \mathbb{P}(P \cup D) = 1 - 0.95 = \boxed{0.05 = 5\%}$$

Problem 3: Soccer

Two soccer teams, the *Gauchos* and the *Bruins*, of equal level are pitted to play against each other. Because the two teams are of equal level, it is assumed that the probability that the *Gauchos* win a game is equal to the probability that the *Bruins* win a game. There is, however, a 10% chance that any given game will result in a tie. Consider the outcome of a single game.

a. What is the outcome space of this experiment?

Solution: At the end of each game, there are only three possibilities: either the *Gauchos* win, the *Bruins* win, or the game results in a tie. As such,

$$\boxed{\Omega = \{\textit{Gauchos} \text{ win}, \ \textit{Bruins} \text{ win}, \ \text{draw}\}}$$

b. What is the probability that the *Gauchos* will win a given game?

Solution: Let p denote the probability that the *Gauchos* win. Then, the probability that the *Bruins* win is also p. Additionally, the probability of a draw is 0.1; hence, by the second axiom of probability (i.e. the fact that the probability of Ω must be 1), we have

$$p + p + 0.1 = 1 \iff 2p = 0.9 \iff \boxed{p = 0.45 = 45\%}$$

Problem 4: Hackers

Congratulations- you've turned to the dark side and are now a hacker! Your first order of business is to try and gain access to people's bank accounts on the *GauchoBank* website.

a. Suppose that passwords on the *GauchoBank* website are of the form: 5 letters (A through Z), followed by 3 digits (0 through 9), followed by 1 special character (!, @, #, $, or %). How many passwords are possible using this scheme?

Solution: Note that our experiment consists of $5 + 3 + 1 = 9$ stages, one for each character in the password. As such, the slot diagram for this experiment looks like:

$$\underline{26} \times \underline{26} \times \underline{26} \times \underline{26} \times \underline{26} \times \underline{10} \times \underline{10} \times \underline{10} \times \underline{5} = (26)^5 \times (10)^3 \times (5)$$

which, when simplified, yields a total of 59,406,880,000 possible passwords.

b. You now find out that the password scheme is a little more complex than you originally thought: specifically, though the letters digits and special characters must remain consecutive, the order in which they appear *across these categories* is not fixed. That is, both `abcde12345%` and `%abcde12345` are valid passwords. How many passwords are possible using this scheme?

Solution: This is very similar to Worked-Out Example 1 from Lecture04. If we fix the order of the categories (i.e. letters, numbers, and special characters) the number of configurations is equal to the answer we found in part (a). However, we need to multiply by the number of ways to arrange the categories amongst themselves- since there are 3 categories, this is equivalent to 3! and hence our final answer is

$$3! \times (26)^5 \times (10)^3 \times (5)$$

which simplifies to 356,441,280,000

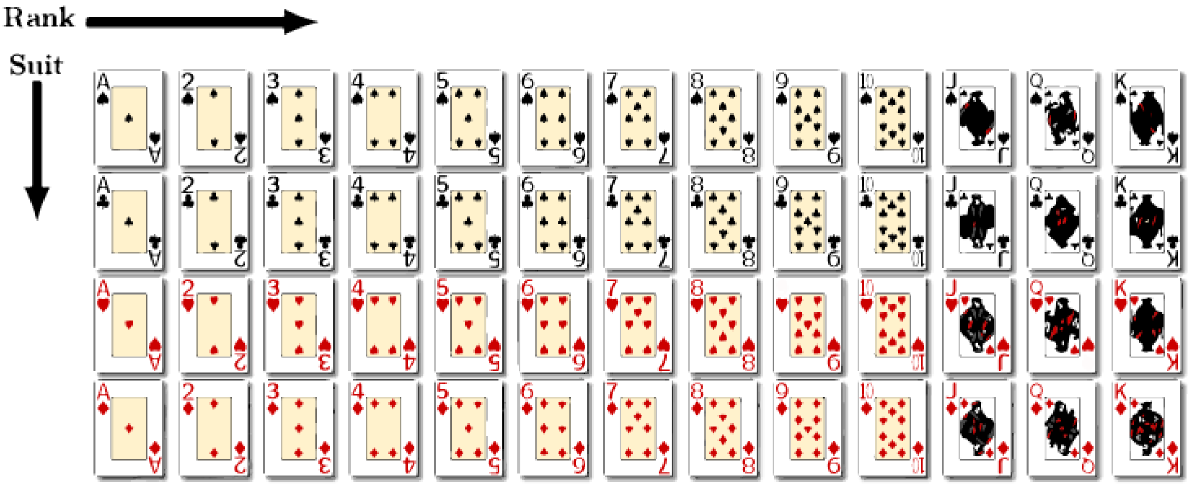
c. Now, suppose that passwords are constructed using the scheme outlined in part (b) above, and suppose you are now interested in hacking into Ethan's *GauchoBank* account. As such, you guess a password at random- what is the probability that you correctly guess Ethan's password? Justify your answer.

Solution: The experiment of picking a password at random has outcome space equal to all the possible passwords; hence, the number of elements in Ω is equal to the answer we found in part (b) above. Because we are picking a password *at random*, we can use the classical approach to probability; as Ethan has only **1** password, the probability of correctly guessing his password is

$$\frac{1}{3! \times (26)^5 \times (10)^3 \times (5)} = \frac{1}{356{,}441{,}280{,}000}$$

Problem 5: Counting Cards

A standard deck of cards contains 52 cards arranged into 4 **suits** (`spades`, `clubs`, `hearts`, and `diamonds`) and 13 **ranks** (`A`, 1 through `10`, `Jack`, `Queen`, and `King`):



a. If a card is drawn at random from a deck of 52 cards, what is the probability that it is a `heart`?

Solution: Let H denote the event "the card selected was a heart". There are 13 cards of the "heart" suit, meaning H has 13 elements in it. Because we are drawing the card *at random*, we can use the classical approach to probability to conclude

$$\mathbb{P}(H) = \frac{13}{52} = \frac{1}{4} = 25\%$$

b. If a card is drawn at random from a deck of 52 cards, what is the probability that it is an Ace?

Solution: Let A denote the event "the card selected was an Ace". There are only 4 Aces in the deck, meaning A has 4 elements in it- by the classical approach to probability, we therefore have

$$\mathbb{P}(A) = \frac{4}{52} = \frac{1}{13} \approx 7.69\%$$

c. If a card is drawn at random from a deck of 52 cards, what is the probability that it is either a King or an Ace?

Solution: Let K denote the event "the card selected was a King" and A denote "the card selected was an Ace." Since K and A are disjoint events, we have that

$$\mathbb{P}(K \cup A) = \mathbb{P}(K) + \mathbb{P}(A) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13} \approx 15.38\%$$

d. If a card is drawn at random from a deck of 52 cards, what is the probability that it is either a King or a spade?

Solution: Let K denote "the card selected was a King" and S denote "the card selected was a spade". Now K and S are no longer disjoint- specifically, $K \cap H$ denotes drawing a card that is both a king and a spade, of which there is only one (i.e. the King of Spades). Hence, by the Addition Rule,

$$\mathbb{P}(K \cup S) = \mathbb{P}(K) + \mathbb{P}(S) - \mathbb{P}(K \cap S) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13} \approx 30.77\%$$

Problem 6: Programming

Instructions

- Write your answers to this question in a new Jupyter notebook, and export your work to a PDF using the steps you saw in Lab01. **Be sure to merge this PDF with your PDF containing your work to the above questions before submitting!** (See instructions at the top of this homework).

Part (a): Create the following table in Python, and assign it to a variable called `star_trek`. Display the `star_trek` table by creating an running a cell containing just the code `star_trek`.

Name	Rank	Species	Homeworld
Kirk	Admiral	Human	Earth
Spock	Commander	Vulcan	Vulcan
Worf	Lt. Commander	Klingon	Q'onoS
Data	Lt. Commander	Android	Omicron Theta
Riker	Commander	Human	Earth

Part(b) Write code to extract the `Rank` column from the `star_trek` table.

Part(c) Write code to count the number of humans that are present in the `star_trek` table. As a hint, you can use the following template and fill in the blanks with appropriate methods:

```
_____.where("Species", _____)._____
```

As a further hint: remember that http://www.data8.org/datascience/tables.html contains a list of methods for tables.

Part (d) Add the following row to the `star_trek` table **WITHOUT** using the `.append()` method:

Name	Rank	Species	Homeworld
Sisko	Captain	Human	Earth

Display the modified `star_trek` table to ensure the addition was successful. **Hint:** There are two things you'll need to do: you'll need to first find a method to add a row to a table that isn't the `.append()` method, and you'll also need to perform variable re-assignment (recall when we did that on Lab1!)

Part (e) What is the probability that a randomly selected person from the `star_trek` table is a human from Earth? **Answer this question AFTER completing the addition of the row in part (d) above.** You don't need to use code for this part, but you should write your answer in a markdown cell.

HW02 Problem 6

PSTAT 5A, compiled by Ethan

April 14, 2023

[1]:
```
from datascience import * # recall that the Table() function is included in the␣
↪datascience module!
```

Part (a)

[2]:
```
star_trek = Table().with_columns(
    "Name", ["Kirk", "Spock", "Worf", "Data", "Riker"],
    "Rank", ["Admiral", "Commander", "Lt. Commander", "Lt. Commander",␣
↪"Commander"],
    "Species", ["Human", "Vulcan", "Klingon", "Android", "Human"],
    "Homeworld", ["Earth", "Vulcan", "Q'onoS", "Omicron Theta", "Earth"]
)
```

[3]:
```
star_trek  # checking to make sure the table looks good
```

[3]:

Name	Rank	Species	Homeworld
Kirk	Admiral	Human	Earth
Spock	Commander	Vulcan	Vulcan
Worf	Lt. Commander	Klingon	Q'onoS
Data	Lt. Commander	Android	Omicron Theta
Riker	Commander	Human	Earth

Part (b)

[4]:
```
star_trek.select("Rank")
```

[4]:

Rank
Admiral
Commander
Lt. Commander
Lt. Commander
Commander

Part (c)

[5]:
```
star_trek.where("Species", "Human").num_rows
```

[5]: 2

Part(d)

```
[6]: star_trek = star_trek.with_row(
         ["Sisko", "Captain", "Human", "Earth"]
     )
```

```
[7]: star_trek  # checking to make sure the table looks good
```

```
[7]: Name  | Rank          | Species | Homeworld
     Kirk  | Admiral       | Human   | Earth
     Spock | Commander     | Vulcan  | Vulcan
     Worf  | Lt. Commander | Klingon | Q'onoS
     Data  | Lt. Commander | Android | Omicron Theta
     Riker | Commander     | Human   | Earth
     Sisko | Captain       | Human   | Earth
```

Part(e)

Since there are 6 people in the dataset, of which only 3 are human, the probability of selecting a human is simply 3/6, or, equivalently, 1/2.