PSTAT 5A: Quiz 02

Summer Session A 2023, with *Ethan P. Marzban*

WRITE YOUR ANSWERS ON A SEPARATE SHEET OF PAPER; DO NOT TRY TO WRITE THEM ON THIS PDF

1. The following two parts are unrelated. Be sure to show all of your steps; if you need to use a lookup table, be sure to explicitly state where in your solutions you used the table.

   (a) (3 points) If $X \sim \mathcal{N}(3,\ 2)$, compute $\mathbb{P}(X \leq 1)$.

   Solution: We first standardize, or, equivalently, find the $z-$score:

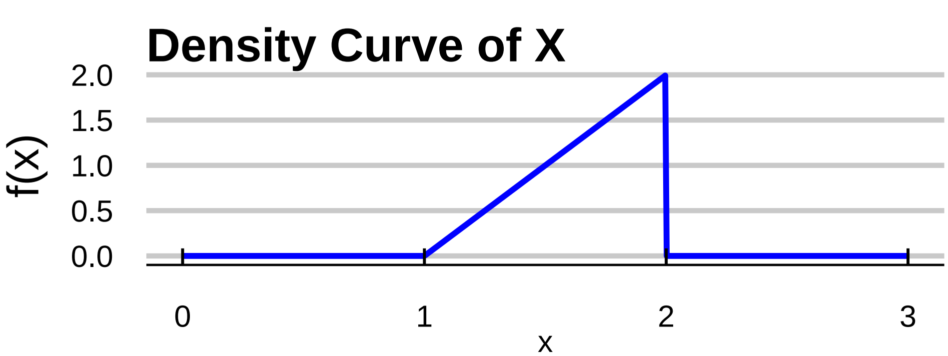
   $$z = \frac{1-3}{2} = -1$$

   Hence, we look up the probability associated with -1.00 in our $z-$table, which gives us a final answer of 0.15870.

   (b) (3 points) If $Z \sim \mathcal{N}(0,\ 1)$, what is the value of c such that $\mathbb{P}(X > c) = 0.7123$?

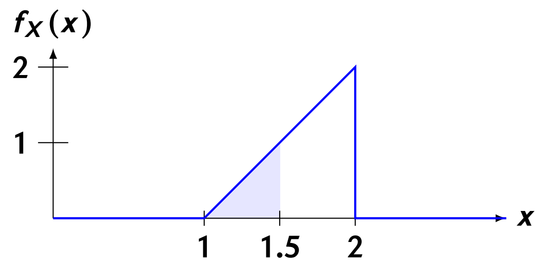
   Solution: If $\mathbb{P}(X > c) = 0.7123$, then, by the Complement Rule, $\mathbb{P}(X \leq c) = 1 - 0.7123 = 0.2877$ meaning c is the 28.77^{th} percentile of the standard normal distribution. Hence, we find the value of 0.2877 in the normal table, and see what $z-$score is associated with that value- this givese us $c = -0.56$.

2. The random variable X has the following density curve (if the picture is difficult to read, the density curve is zero up to 1, a straight line from the point $(1, 0)$ to $(2, 2)$, and then zero from 2 onwards):



(a) (3 points) Let $F_X(x)$ denote the cumulative distribution function (c.d.f.) of X at x. What is the value of $F_X(1.5)$? Remember to sketch a picture for full credit!

Solution: By definition, $F_X(x) = \mathbb{P}(X \leq x)$ meaning $F_X(1.5) = \mathbb{P}(X \leq 1.5)$ which can be found as the area of the following region:

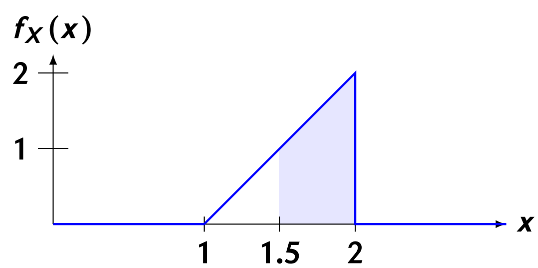


This is a triangle with base $(1.5-1) = 0.5$ and height 1; hence, its area - and, consequently, the desired probability - is

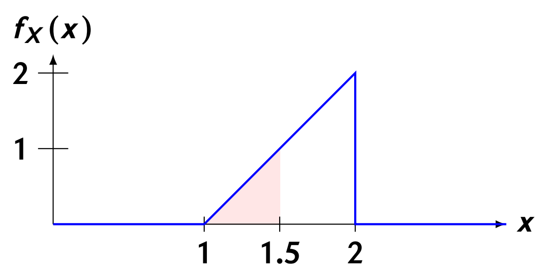
$$\frac{1}{2}(1.5 - 1)(1) = \frac{1}{2} \cdot \frac{1}{2} = \boxed{\frac{1}{4}}$$

(b) (4 points) What is $\mathbb{P}(1.5 \leq X \leq 3)$? Remember to sketch a picture for full credit!

Solution: We, again, sketch a picture: this time, we need to be careful about the region we are sketching, as the state space of X only extends to 2. As such, the region of interest is:



We could try to find the area of this trapezoid directly (which would be a perfectly correct approach), but it will be easier to simply compute its area as one minus the area of the following red triangle:



The area of this red triangle is precisely what we found in part (a), and is therefore equal to $(1/4)$: hence,

$$\mathbb{P}(1.5 \leq X \leq 3) = 1 - \frac{1}{4} = \frac{3}{4}$$

3. In a large orchard, 65% of trees are orange trees. A sample of 13 trees is taken, with replacement, and the number of these trees that are orange trees is recorded.

(a) (1 point) Define the random variable of interest, and call it X.

Solution: Let X denote the number of trees, in the sample of 13, that are orange trees.

(b) (4 points) What is the distribution of X? Be sure to include any/all parameter(s), and check all relevant conditions!

Solution: We surmise that X is Binomially distributed; to verify this, we check the four Binomial conditions:

1) **Independent Trials?** Yes, since sampling is done with replacement.
2) **Fixed Number of Trials?** Yes, $n = 13$
3) **Well-defined notion of Success?** Yes; 'success' = 'tree produces fruit'.
4) **Fixed probability of Success?** Yes; $p = 0.65$

Since all four conditions are met, we conclude that

$$X \sim \text{Bin}(13,\ 0.65)$$

(c) (3 points) What is the probability that exactly 8 of these 13 trees are orange trees? You do not need to simplify your answer to a decimal, but be sure to show all of your work!

Solution: We seek $\mathbb{P}(X = 8)$. Using the formula for the probability mass function of the Binomial Distribution, we have

$$\mathbb{P}(X = 8) = \binom{13}{8}(0.65)^8(1 - 0.65)^{13-8} \approx 0.2154$$