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PSTAT 5A: Discussion Worksheet 03

Summer Session A 2023, with *Ethan P. Marzban*

1. Consider the following game: a fair six-sided die is rolled. If the number showing is '1' or '2', you win a dollar; if the number showing is '3', '4', or '5' you win 2 dollars; if the number showing is '6', you lose 1 dollar. Let W denote your net winnings after playing this game once.

   (a) Write down the state space S_W of W.

   Solution: We know that our net winnings will either be +1, +2, or −1; therefore,

   $$S_W = \{-1,\ 1,\ 2\}$$

   (b) Find the p.m.f. of W. **Hint:** You may want to define a new variable to keep track of the result of the die roll.

   Solution: Following the hint, define X to be the result of the die roll. Our goal is therefore to translate $\mathbb{P}(W = k)$ (for $k = -1,\ 1,\ 2$) to probabilities in terms of X.

   - We only lose a dollar if the die lands on '6'; as such,

     $$\mathbb{P}(W = -1) = \mathbb{P}(X = 6) = \frac{1}{6}$$

     where we have utilized the Classical Approach to Probability (which we are allowed to use since the die is <u>fair</u>) to compute the probability that the die lands on '6' to be $1/6$.

   - We only gain a dollar if the die lands on '1' or '2'; as such,

     $$\mathbb{P}(W = 1) = \mathbb{P}(\{X = 1\} \cup \{X = 2\}) = \mathbb{P}(X = 1) + \mathbb{P}(X = 2) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

   - We only gain two dollars if the die lands on '3', '4', or '5'; as such,

     $$\begin{aligned} \mathbb{P}(W = 2) &= \mathbb{P}(\{X = 3\} \cup \{X = 4\} \cup \{X = 5\}) \\ &= \mathbb{P}(X = 3) + \mathbb{P}(X = 4) + \mathbb{P}(X = 5) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2} \end{aligned}$$

Therefore, putting everything together, we obtain the following p.m.f. for W:

k	-1	1	2
$\mathbb{P}(W = k)$	$1/6$	$1/3$	$1/2$

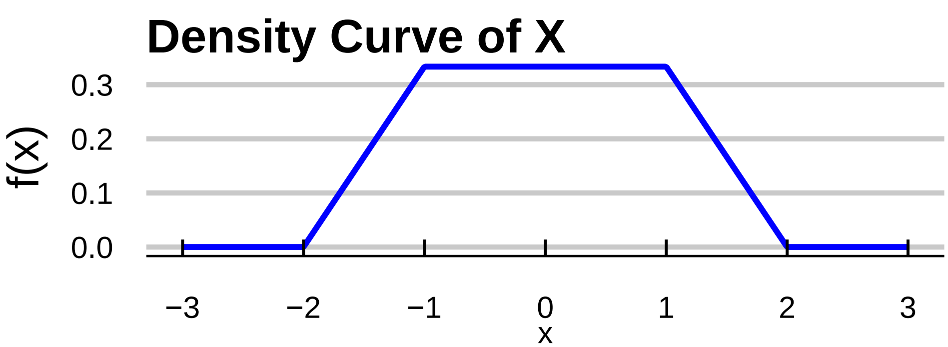
As a quick sanity check, $(1/6) + (1/3) + (1/2) = 1$, as exepcted.

(c) What are your expected winnings after one round of the game?

Solution:

$$\begin{aligned}\mathbb{E}[W] &= \sum_{\text{all } k} k \cdot \mathbb{P}(W = k) \\ &= (-1) \cdot \mathbb{P}(W = -1) + (1) \cdot \mathbb{P}(W = 1) + (2) \cdot \mathbb{P}(W = 2) \\ &= (-1) \cdot \left(\frac{1}{6} \right) + (1) \cdot \left(\frac{1}{3} \right) + (2) \cdot \left(\frac{1}{2} \right) = \frac{7}{6} = 1.1\overline{6}\end{aligned}$$

2. The random variable X has the following density curve (in case it is difficult to read, $f_X(x)$ attains a constant value of $1/3$ for $-1 \leq x \leq 1$):



(a) What is S_X, the state space of X?

Solution: The state space of a continuous random variable can be seen to be the set of values for which the density curve is nonzero. From this graph, we can therefore infer that

$$S_X = [-2, 2]$$

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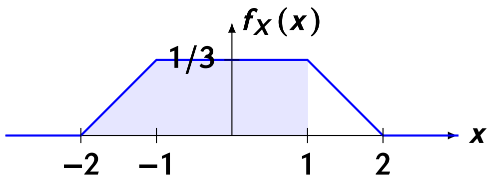
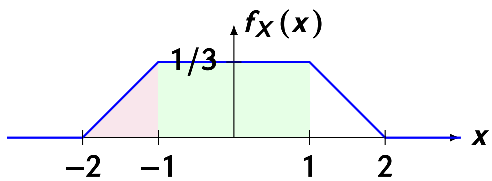
(b) Let $F_X(x)$ denote the cumulative distribution function (c.d.f.) of X at x. What is the value of $F_X(1)$?

Solution: Definitionally, $F_X(x) = \mathbb{P}(X \leq x)$. Hence, $F_X(1) = \mathbb{P}(X \leq 1)$, which we find as the following area:



There are three ways we can find this area.

Method 1: The first is to treat it as the sum of the area of a rectangle and the area of a triangle:

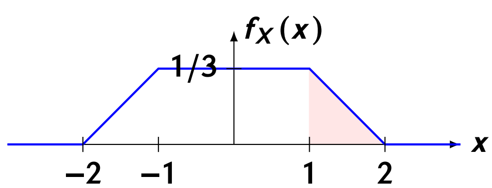


$$\implies \frac{1}{2}[-1-(-2)]\left(\frac{1}{3}\right) + [1-(-1)]\left(\frac{1}{3}\right) = \frac{5}{6}$$

Method 2: Another way is to use the formula for the area of a trapezoid, with $a = [1-(-2)] = 3,\ b = [1-(-1)] = 2$, and $h = 1/3$:

$$\implies \frac{a+b}{2} \cdot h = \frac{3+2}{2} \cdot \frac{1}{3} = \frac{5}{6}$$

Method 3: Another way to obtain this area is to use the complement rule: since the area under the full density curve is 1, the desired area can be found as 1 minus the area of the triangle below:



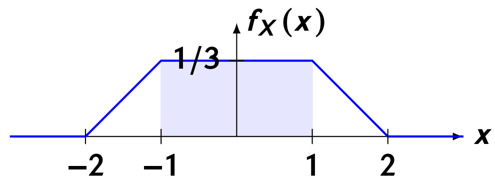
$$\implies 1 - \frac{1}{2}(2-1)\left(\frac{1}{3}\right) = 1 - \frac{1}{6} = \frac{5}{6}$$

Regardless, we obtain a final answer of

$$F_X(1) = \frac{5}{6} = 0.8\overline{3}$$

(c) What is $\mathbb{P}(-1 \leq X \leq 1)$?

Solution: Again, we sketch a picture:

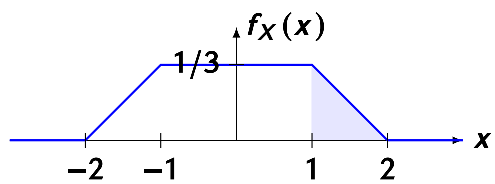


The area of this rectangle, and consequently the desired probability, is

$$[1 - (-1)] \cdot \left(\frac{1}{3} \right) = \boxed{\frac{2}{3} = 0.\overline{6}}$$

(d) What is $\mathbb{P}(1 \leq X \leq 3)$?

Solution: As always, we sketch a picture. However, we need to be a bit careful: note that the trapezoidal shape of the density curve does *not* extend out to $x = 3$, as the largest value in the state space of X is 2. As such, the region we sketch is:



The area of this triangle, and consequently the desired probability, is

$$\frac{1}{2}(2 - 1) \left(\frac{1}{3} \right) = \boxed{\frac{1}{6} = 0.1\overline{6}}$$

3. The time a randomly-selected person will spend waiting in line for the new hit restaurant *GauchoYum* varies according to a normal distribution with mean 15 minutes and standard deviation 4 minutes. Suppose a person is selected at random, and the amount of time (in minutes) they spend waiting in line is recorded.

(a) Define the random variable of interest, and call it X.

Solution: Let X denote the time (in minutes) a randomly-selected person spends waiting in line at *GauchoYum*.

(b) Using proper notation, write down the distribution of X.

Solution: $X \sim \mathcal{N}(15,\ 4)$

(c) What is the probability that this randomly-selected person will wait exactly 12 minutes in line?

Solution: We seek $\mathbb{P}(X = 12)$. Since X is continuous, we know that $\mathbb{P}(X = k) = 0$ for any value of k; hence, the desired probability is $\boxed{0}$.

(d) What is the probability that this randomly-selected person will wait less than 16 minutes?

Solution: We wish to compute $\mathbb{P}(X \leq 16)$. We do so by standardizing, and then using our normal table:

$$\mathbb{P}(X \leq 16) = \mathbb{P}\left(\frac{X - 15}{4} \leq \frac{16 - 15}{4} \right) = \mathbb{P}\left(\frac{X - 15}{4} \leq 0.25 \right)$$

Our Standardization Result tells us that $[(X - 15)/4] \sim \mathcal{N}(0,\ 1)$; hence, the probability above can be found by looking up the value of 0.25 in our normal table. This yields a final result of $\boxed{0.5987}$.

(e) What is the probability that this randomly-selected person will wait between 12 and 17 minutes?

Solution: We wish to compute $\mathbb{P}(12 \leq X \leq 17)$. Our familiar procedure tells us to first write this as a difference of left-tail probabilities:

$$\mathbb{P}(12 \leq X \leq 17) = \mathbb{P}(X \leq 17) - \mathbb{P}(X \leq 12)$$

Now, we compute each of the probabilities on the RHS by again standardizing and using our lookup table:

$$\begin{aligned} \mathbb{P}(X \leq 17) - \mathbb{P}(X \leq 12) &= \mathbb{P}\left(\frac{X - 15}{4} \leq \frac{17 - 15}{4} \right) - \mathbb{P}\left(\frac{X - 15}{4} \leq \frac{12 - 15}{4} \right) \\ &= \mathbb{P}\left(\frac{X - 15}{4} \leq 0.5 \right) - \mathbb{P}\left(\frac{X - 15}{4} \leq -0.75 \right) \\ &= 0.6915 - 0.2266 = \boxed{0.4649} \end{aligned}$$

(f) Suppose now that a representative sample of 10 people is taken with replacement, and the number of people (in this group of 10) that wait between 12 and 17 minutes is recorded. What is the probability that precisely 4 of these 10 people wait between 12 and 17 minutes? **Hint:** You may need to define a new random variable and identify its distribution; if you do so, be sure to check any/all relevant conditions!

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Solution: Let Y denote the number of people, in a sample (taken with replacement) of 10, that wait between 12 and 17 minutes in line. We surmise that Y is Binomially distributed; to verify this, we check the four Binomial Criteria:

1) **Independent trials?** Yes, since sampling is done with replacement.

2) **Fixed number of trials?** Yes; $n = 10$ trials.

3) **Well-defined notion of 'success'?** Yes; 'success' = 'waiting in line for between 12 and 17 minutes'

4) **Fixed probability of success?** Yes; $p = 0.4649$, as found in part (e) above.

Since all 4 conditions are satisfied, we conclude that $Y \sim \text{Bin}(10,\ 0.4649)$.

Now, we seek $\mathbb{P}(Y = 4)$, which we can now compute using the formula for the p.m.f. of the Binomial distribution:

$$\mathbb{P}(Y = 4) = \binom{10}{4}(0.4649)^4(1 - 0.4649)^{10-4} \approx 0.2302$$